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13. ABSTRACT

Three different notes are presented here which are related to certain new and simple concepts of non-cooperative n-person games. These are natural generalizations of the notions of maximin and minimax strategies and the saddle points of two-person games. The concept of the equilibrium point appears as a special case of one of these.

The first note expresses some intuitive considerations for games on Euclidean spaces. Their characterizations are essentially given by Kakutani's fixed point theorem. As a particular case, we examine such points for the mixed extensions of finite n-person games.

The second and third notes are concerned with two different mathematical extensions of the results obtained in the first note. They are based respectively on Fan's and Nikaido-Isoda's ideas of proving the existence of equilibrium points for games on real linear topological spaces. In particular, the concepts introduced in the first note are examined for mixed extensions of continuous games. These last two notes involve the use of more advanced mathematical techniques than does the first.

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# "SIMPLE" STABILITY OF GENERAL n-PERSON GAMES

Ezio Marchi

Econometric Research Program
Research Memorandum No. 84
February 1967

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Three different notes are presented here which are related to certain new and simple concepts of non-cooperative n-person games. These are natural generalizations of the notions of maximin and minimax strategies and the saddle points of two-person games. The concept of the equilibrium point appears as a special case of one of these.

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# "SIMPLE" STABILITY OF GENERAL n-PERSON GAMES

Ezio Marchi

I. Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be an n-person game in normal form and $N = \{1,\ldots,n\}$ the set of players where for player $i\epsilon N$, $\Sigma_i$ is the strategy set, assumed to be non-empty, compact and convex in a Euclidean space, and $A_i$ a real function on $\Sigma_N = \underset{i\epsilon N}{X} \Sigma_i$ is the payoff. A function $\underline{e}: N \to P_N$, where $P_N$ denotes the set of all subsets of $N$, is said to be a simple structure function of the game $\Gamma$ if for all $i\epsilon N$ the set $e(i)$ is included in the set $N - \{i\}$. We define $\Gamma_{\underline{e}} = (\Gamma,\underline{e})$ as the game $\Gamma$ with simple structure function $\underline{e}$, and to simplify, $\Gamma_{\underline{e}}$ is said to be a game. For the player $i\epsilon N$ belonging to the game $\Gamma_{\underline{e}}$, the sets $e(i)$ and $f(i) = N-(e(i)\cup\{i\})$ are the antagonistic and the indifferent coalitions respectively. The strategy $\sigma\epsilon\Sigma_N$ can be represented for $i\epsilon N$ as $\sigma = (\sigma_i,\sigma_{e(i)},\sigma_{f(i)})$ where $\sigma_i\epsilon\Sigma_i$, and $\sigma_R\epsilon\Sigma_R = \underset{j\epsilon R}{X} \Sigma_j$ where $R$ is $e(i)$ or $f(i)$.

Given the strategy $\sigma\epsilon\Sigma_N$; for $i\epsilon N$ let

$$\Gamma(\sigma_{f(i)}) = \{\Sigma_i,\Sigma_{e(i)}; A_i(\sigma_i,\sigma_{e(i)},\sigma_{f(i)})\}$$

be a zero-sum two person game whose maximin value is

$$v_i(\sigma_{f(i)}) = \max_{s_i\epsilon\Sigma_i} \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(s_i,s_{e(i)},\sigma_{f(i)}) .$$

If $\Gamma_{\underline{e}}$ is a game with a simple structure function $\underline{e}$, then a strategy $\bar{\sigma}\epsilon\Sigma_N$ of the game $\Gamma_{\underline{e}}$ is called an $\underline{e}$-maximin simple stable point, concisely $\underline{e}_m$-simple stable if:

$$\min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)}) = v_i(\bar{\sigma}_{f(i)}) \quad \text{for all } i\in N .$$

A strategy $\bar{\sigma}\in\Sigma_N$ is $\underline{e}_m$-simple stable of the game $\underline{e}$ if and only if it is an equilibrium point of the game $\Gamma^* = [\Sigma_1,\ldots,\Sigma_n; F_1,\ldots,F_n]$ where the payoff $F_i$ is defined by:

$$F_i(\sigma) = \min_{s_{e(i)}\in\Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)}) \quad \text{for } i\in N \text{ and } \sigma\in\Sigma_N .$$

Intuitively speaking, the coalition $e(i)$ of the player $i\in N$ in the game $\Gamma_{\underline{e}}$ is the set of players that can enter into an alliance non cooperatively. Thus the behavior of the members of the coalition can be viewed as directed towards hurting player $i\in N$ in an non-cooperative manner. The coalition $f(i)$ are the indifferent players. An $\underline{e}_m$-simple stable point is a rule of behavior which on the one hand assures at least the amount $v_i(\bar{\sigma}_{f(i)})$ to each player independently on th behavior of the antagonistic coalition and on the other hand such that the value $v_i(\bar{\sigma}_{f(i)})$ is the maximum safety value which the mentioned player is able to get, if in each instance all the players of his indifferent coalition abide by it. The outcome for the palyer $i\in N$ with respect to the strategy $\bar{\sigma}\in\Sigma_N$, $\underline{e}_m$-simple point in the game $\Gamma_{\underline{e}}$ is:

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) \geq v_i(\bar{\sigma}_{f(i)}) .$$

There are two interesting particular cases of $\underline{e}_m$-simple stable points, which characterize extreme structures of games: (i) if each indifferent coalition is empty; and (ii) if each antagonistic coalition is void. In the last case such a point is an equilibrium point.

<u>THEOREM:</u> If for each $i \in N$ the game $\Gamma_{\underline{e}}$ satisfies the following conditions: $\Sigma_i$ is compact and convex set in an Euclidean space; $A_i$ is continuous in $\sigma \in \Sigma_N$ and

$$F_i(\sigma_i, \sigma_{f(i)}) = \min_{s_{e(i)} \in \Sigma_{e(i)}} A_i(\sigma_i, s_{e(i)}, \sigma_{f(i)})$$

is concave with respect to $\sigma_i \in \Sigma_i$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$ ; then $\Gamma_{\underline{e}}$ has at least one $\underline{e}_m$-simple stable point.

<u>**PROOF:**</u> Given $\sigma \in \Sigma_N$ and $i \in N$ , we define the set

$$R_i(\sigma) = \{\tau \in \Sigma_N : F_i(\tau_i, \sigma_{f(i)}) = \max_{s_i \in \Sigma_i} F_i(s_i, \sigma_{f(i)})\} .$$

The function $F_i$ on the compact set $\Sigma_i$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$ , is continuous, since the function $A_i$ is continuous on $\Sigma_N$ and therefore $R_i(\sigma)$ is non-empty. If $\tau^1$ , $\tau^2 \in R_i(\sigma)$ , let $\lambda\tau^1 + (1-\lambda)\tau^2$ be in $\Sigma_N$ where $\lambda \in [0,1]$ . By concavity of the function $F_i$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$ we obtain

$$F_i(\lambda\tau_i^1 + (1-\lambda)\tau_i^2 , \sigma_{f(i)}) \geq \max_{s_i \in \Sigma_i} F_i(s_i, \sigma_{f(i)}) ,$$

consequently the set $R_i(\sigma) \subseteq \Sigma_N$ is convex. Let $\Psi : \Sigma_N \to \Sigma_N$ be a multivalued function defined by $\Psi(\sigma) = \bigcap_{i \in N} R_i(\sigma)$ and let $\sigma(k) \to \sigma$ , $\tau(k) \to \tau$ be two convergent sequences in $\Sigma_N$ , which are such that for each positive integer $k$ : $\tau(k) \in \Psi(\sigma(k))$ . By definition we have for all positive integers $k$ and $i \in N$ : $\tau(k) \in R_i(\sigma(k))$ , i.e.,

$$F_i(\tau_i(k)\ ,\ \sigma_{f(i)}(k)) \ = \ \max_{s_i \in \Sigma_i} F_i(s_i, \sigma_{f(i)}(k))$$

and by continuity of the function $A_i$ :

$$F_i(\tau_i(k)\ ,\ \sigma_{f(i)}(k)) \to F_i(\tau_i, \sigma_{f(i)})$$

and

$$\max_{s_i \in \Sigma_i} F_i(s_i, \sigma_{f(i)}(k)) \ \to \ \max_{s_i \in \Sigma_i} F_i(s_i, \sigma_{f(i)}) \ .$$

Then $\tau \epsilon R_i(\sigma)$ for all $i \epsilon N$ ; so we have obtained $\tau \epsilon \psi(\sigma)$ . Furthermore, the function $\psi$ is upper semi-continuous. We can now apply the fixed-point theorem of Kakutani, since the assumption of this theorem is satisfied for the function $\psi$ , and since the set $\Sigma_N$ is non-empty, compact and convex in a Euclidean space. It follows that there exists a fixed point $\bar{\sigma} \epsilon \Sigma_N$: $\bar{\sigma} \epsilon \psi(\bar{\sigma})$ , which is an $\underline{e}_m$-simple stable point of the game $\Gamma_{\underline{e}}$ . Q.E.D.

If $\Gamma_{\underline{e}}$ is a game with the simple structure function $\underline{e}$ , then a strategy $\bar{\sigma} \epsilon \Sigma_N$ is called an $\underline{e}$-minimax simple point or concisely an $\underline{e}^m$-simple stable point of the game $\Gamma_{\underline{e}}$ if:

$$\max_{s_i \in \Sigma_i} A_i(s_i, \bar{\sigma}_{e(i)}, \ \bar{\sigma}_{f(i)}) \ = \ v^i(\bar{\sigma}_{f(i)}) \qquad \text{for all } i \epsilon N \ ,$$

where

$$v^i(\sigma_{f(i)}) \ = \ \min_{s_{e(i)} \in \Sigma_{e(i)}} \ \max_{s_i \in \Sigma_i} A_i(s_i, s_{e(i)}, \sigma_{f(i)}) \quad \text{for all } i \epsilon N \ ;$$

is the minimax value of the game $\Gamma(\sigma_{f(i)})$ .

Intuitively speaking, an $\underline{e}^m$-simple stable point is a rule of behavior which on the one hand assures to each antagonistic coalition that its corresponding player cannot safely obtain more than $v^i(\bar{\sigma}_{f(i)})$, independent of his own behavior and on the other hand such that the value is the maximum value that the antagonistic coalition will be able to safely prevent against its corresponding player's behavior if in each instance all the players of his indifferent coalition abide by it. The outcome for the player $i \epsilon N$ with respect to the strategy $\bar{\sigma} \epsilon \Sigma_N$, $\underline{e}^m$-simple stable point in the game $\Gamma_{\underline{e}}$, is:

$$A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) \leq v^i(\bar{\sigma}_{f(i)}) \quad .$$

If every antagonist coalition is empty in the game $\Gamma_{\underline{e}}$, then each strategy $\sigma \epsilon \Sigma_N$ is an $\underline{e}^m$-simple stable point. Another extreme case appears when every indifferent coalition is empty.

<u>THEOREM:</u> If for each $i \epsilon N$ the game $\Gamma_{\underline{e}}$ satisfies the following conditions: $\Sigma_i$ is compact and convex set in an Euclidean space; $A_i$ is continuous in $\sigma \epsilon \Sigma_N$;

$$G_i(\sigma_{e(i)}, \sigma_{f(i)}) = \max_{s_i \epsilon \Sigma_i} A_i(s_i, \sigma_{e(i)}, \sigma_{f(i)})$$

is convex with respect to $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$.
Then if for $\sigma \epsilon \Sigma_N$ there is a $\tau \epsilon \Sigma_N$ such that:

$$G_i(\tau_{e(i)}, \sigma_{f(i)}) = v^i(\sigma_{f(i)}) \quad \text{for each} \quad i \epsilon N ,$$

$\Gamma_{\underline{e}}$ has at least one $\underline{e}^m$-simple stable point.

<u>PROOF:</u> Given $\sigma \epsilon \Sigma_N$ and $i \epsilon N$, we define the following non-empty set

$$S_i(\sigma) = \{\tau \epsilon \Sigma_N \; ; \; G_i(\tau_{e(i)}, \sigma_{f(i)}) = \min_{s_{e(i)} \epsilon \Sigma_{e(i)}} G_i(s_{e(i)} \sigma_{f(i)}) \} \; .$$

If $\tau^1, \tau^2 \in S_i(\sigma)$ [illegible] $\lambda\tau^1 + (1-\lambda)\tau^2$ is in $\Sigma_i$ where $\lambda \in [0,1]$. By the

convexity [illegible] $G_i$ [illegible] we obtain

$$G_i(\lambda\tau^1_{s(i)} + (1-\lambda)\tau^2_{s(i)}, [\text{illegible}]) \leq \min_{\tau_{s(i)} \in \Sigma_{s(i)}} G_i(\tau_{s(i)}, \sigma_{r(i)}),$$

and [illegible] $\sigma \in \Sigma_i$ and set $S_i(\sigma) \subseteq \Sigma_i$ is convex. Let $F : \Sigma_N \to \Sigma_N$

be a multi-valued function [illegible] $\bigcap_{i \in N} S_i(\sigma)$, with regard to last

condition [illegible] $\sigma \in \Sigma_i$ [illegible]. Let $\sigma(k) \to \sigma$ and $\tau(k) \to \tau$

be [illegible] sequences in $\Sigma_i$, such that for each positive integer $k$ :

$\tau(k) \in F(\sigma(k))$. [illegible], we have that for all integers $k$ and $i \in N$ :

$\tau(k) \in S_i(\sigma(k))$, i.e., :

$$G_i([\text{illegible}]) = \min_{\tau_{s(i)} \in \Sigma_{s(i)}} G_i(\tau_{s(i)}, \sigma_{r(i)}(k)).$$

By continuity [illegible] :

$$G_i([\text{illegible}]) \to G_i(\tau_{s(i)}, \sigma_{r(i)})$$

and

$$\min_{\tau_{s(i)} \in \Sigma_{s(i)}} G_i(\tau_{s(i)}, \sigma_{r(i)}(k)) \to \min_{\tau_{s(i)} \in \Sigma_{s(i)}} G_i(\tau_{s(i)}, \sigma_{r(i)}).$$

Hence $\tau \in S_i(\sigma)$ for all $i \in N$ [illegible] $\tau \in F(\sigma)$ which proves the upper

semicontinuity [illegible]. Therefore, by application of Kakutani's theorem

[illegible] $\sigma \in \Sigma_i$ : $\sigma \in F(\sigma)$, which is an $\underline{s}$-simple stable point

of the game $\Gamma$ [illegible]. q.e.d.

[illegible] in the following

way : [illegible] which is such that if all the players of the indifferent coalition of any player abide by the first one, the second one is minimax for his antagonistic coalition in the resulting game.

<u>THEOREM:</u> If for each $i \in N$ the game $\Gamma_{\underline{e}}$ satisfies the following conditions: $\Sigma_i$ is compact and convex in an Euclidean space; $A_i$ is continuous in $\sigma \in \Sigma_N$ ; $F_i(\sigma_i, \sigma_{f(i)})$ is concave with respect to $\sigma_i \in \Sigma_N$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$; and $G_i(\sigma_{e(i)}, \sigma_{f(i)})$ is convex with respect to $\sigma_{e(i)} \in \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$. Then if for each $\sigma \in \Sigma_N$ there is a $\tau \in \Sigma_N$ such that for all $i \in N$:

$$F_i(\tau_i, \sigma_{f(i)}) = v_i(\sigma_{f(i)}) \quad \text{and} \quad G_i(\tau_{e(i)}, \sigma_{f(i)}) = v^i(\sigma_{f(i)}) ,$$

then the game $\Gamma_{\underline{e}}$ has at least one $\underline{e}^m$-simple and $\underline{e}_m$-simple stable point.

<u>PROOF:</u> Let $\Psi : \Sigma_N \to \Sigma_N$ be a multivalued function defined by $\Psi(\sigma) = \bigcap_{i \in N} [S_i(\sigma)\ R_i(\sigma)]$ where $S_i(\sigma)$ and $R_i(\sigma)$ have been defined previously. For each $\sigma \in \Sigma_N$ , the set $\Psi(\sigma)$ is non-empty and convex. By the continuity of the function $A_i$ , the function $\Psi$ is upper-semicontinuous, then applying Kakutani's theorem, the existence of a fixed point $\bar{\sigma} \in \Sigma_N : \bar{\sigma} \in \Psi(\bar{\sigma})$ is guaranteed. Such a strategy is $\underline{e}^m$-simple and $\underline{e}_m$-simple stable point. Q.E.D.

For any established behavior among the players, there is another one which is such that, if all the players of the indifferent coalition of any player abide by the first one, the second one is minimax for his antagonistic coalition and maximin for himself in the resulting game. Such is a possible interpretation to the last condition in the above theorem. The outcome of player $i \in N$ with respect to the strategy $\bar{\sigma} \in \Sigma_N$ $\underline{e}^m$-simple and $\underline{e}_m$-simple stable in the game $\Gamma_{\underline{e}}$ is

$A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)})$ which satisfies:

$$v_i(\bar{\sigma}_{f(i)}) \leq A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) \leq v_i(\bar{\sigma}_{f(i)})$$

Such a point we call e-simple stable. An e-simple point is a rule of behavior which is maximin for each player and minimax for his antagonistic coalition in the resulting game, if in each instance all the players of his indifferent coalition abide by it.

An interesting particular case of the e-simple stable point appears when the above relations hold as equalities. An immediate result is given in the following:

THEOREM: If for each $i \epsilon N$ the game $\Gamma_e$ satisfies the following conditions: $\Sigma_i$ is compact and convex in an Euclidean space, $A_i$ is continuous in $\sigma \epsilon \Sigma_N$, concave with respect $\sigma_i \epsilon \Sigma_i$ for fixed $(\sigma_{e(i)}, \sigma_{f(i)}) \epsilon \Sigma_{e(i)} X \Sigma_{f(i)}$ and convex with respect $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$ for fixed $(\sigma_i, \sigma_{f(i)}) \epsilon \Sigma_i X \Sigma_{f(i)}$.

Then if for each $\sigma \epsilon \Sigma_N$ there is a $\tau \epsilon \Sigma_N$ such that

$$F_i(\tau_i, \sigma_{f(i)}) = v_i(\sigma_{f(i)}) \quad \text{and} \quad G_i(\tau_{e(i)}, \sigma_{f(i)}) = v^i(\sigma_{f(i)})$$

then the game $\Gamma_e$ has a e-simple stable point $\bar{\sigma} \epsilon \Sigma_N$:

$$v_i(\bar{\sigma}_{f(i)}) = A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = v_i(\bar{\sigma}_{f(i)}) \quad \text{for all } i \epsilon N$$

PROOF: The first conditions assure for each $i \epsilon N$ the concavity of the function $F_i(\sigma_i, \sigma_{f(i)})$ in $\sigma_i \epsilon \Sigma_i$ for fixed $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$ and the convexity of the function $G_i(\sigma_{e(i)}, \sigma_{f(i)})$ in $\sigma_{e(i)} \epsilon \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \epsilon \Sigma_{f(i)}$.

Therefore, by the above theorem the existence of an e-simple stable point $\bar{\sigma} \epsilon \Sigma_N$ is guaranteed, and the theorem is proved since for each $i \epsilon N$ and $\sigma \epsilon \Sigma_N$ we have $v_i(\sigma_{f(i)}) = v^i(\sigma_{f(i)})$. Q.E.D.

Such a point we call an n-person e-simple saddle point or rather a e-simple saddle point of the game $\Gamma_e$. An e-simple saddle point is a rule of behavior which for each player $i \epsilon N$ is saddle point in the resulting game, if all the players of the indifferent coalition abide by it. In other words, it is optimal for each player and each antagonistic coalition, given the actions of the indifferent coalitions. As a simple illustration, let $\Gamma_e$ be the mixed extension of a finite two person game $\Gamma_e$ with its structure function defined by: $e(1) = \{2\}$, $e(2) = \{1\}$. For this game the existence of an e-simple saddle point is equivalent to the following condition: $U_1 \cap V_2$ and $U_2 \cap V_1$ are non-empty, where $U_i$ is the set of maximin strategies for player $i$ and $V_j$ is the set of minimax strategies for player $j \neq i$ in the zero-sum two-person game $\tilde{\Gamma}_i = \{\tilde{\Sigma}_i, \tilde{\Sigma}_{j\neq i} ; A_i\}$ where $i,j = 1,2$.

II. In this section, we examine some applications that deal with finite games. We need the following:

LEMMA: The mixed extension $\tilde{\Gamma}_e = \{\tilde{\Sigma}_1,\ldots,\tilde{\Sigma}_n; E_1,\ldots,E_n\}$ of a finite n-person game $\Gamma_e = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ with simple structure function $e$ such that for each $i \epsilon N$ and each $(x_i, x_{f(i)}) \epsilon X_i \times X_{f(i)}$ the function $E_i(X_i, X_{e(i)}, X_{f(i)})$ is linear in $x_{e(i)} \epsilon X_{e(i)}$; for each $i \epsilon N$ satisfies the following: $E_i$ is continuous in the variable $x \epsilon X_N = \underset{i \epsilon N}{\times} \tilde{\Sigma}_i$;

$$v_i(x_{f(i)}) = v^i(x_{f(i)}) \text{ for each } x_{f(i)} \epsilon X_{f(i)} = \underset{j \epsilon f(i)}{\times} \tilde{\Sigma}_j ,$$

where $v_i(x_{f(i)})$ and $v^i(x_{f(i)})$ are the respectives maximin and minimax values of the zero-sum two-person game

$$\tilde{\Gamma}(x_{f(i)}) = \{X_i, X_{e(i)}; E_i(x_i, x_{e(i)}, x_{f(i)})\} .$$

**PROOF:** For $i \in N$, the function $E_i$ is continuous with respect to $x \in X$ it is a multilinear function. The function $E_i$ is continuous with respect to variable $(x_i, x_{e(i)})$ in $X_i \times X_{e(i)}$ which is compact and convex. The

$$\min_{\omega_{e(i)} \in X_{e(i)}} E_i(x_i, \omega_{e(i)}, x_{f(i)})$$

is concave in $x_i \in X_i$ for fixed $x_{f(i)} \in X_{f(i)}$ and the function

$$- \max_{\omega_i \in X_i} E_i(\omega_i, x_{e(i)}, x_{f(i)}) \text{ is also concave in } x_{e(i)} \in X_{e(i)} \text{ for fixed}$$

$x_{f(i)} \in X_{f(i)}$. Then the game $\tilde{\Gamma}(x_{f(i)})$ has an equilibrium point and therefore $v_i(x_{f(i)}) = v^i(x_{f(i)})$. Q.E.D.

We note that the strong condition of linearity on the expectation function is nece in the above formulation, since otherwise the equality of the maximin $v_i(x_{f(i)})$ and minimax $v^i(x_{f(i)})$ values is not guaranteed.

This fact is illustrated in the following example:

Given the finite zero-sum two-person game

$$\Gamma = \{\Sigma_1, \Sigma_2 ; A\}$$

where the strategies sets are

$$\Sigma_1 = \Sigma \times \Sigma , \quad \Sigma_2 = \Sigma$$

with $\Sigma = \{1,2\}$ and the payoff defined by

$$A(\sigma_1, \sigma_2, \sigma_3) = \begin{cases} 1 & \text{if } \sigma_1 = \sigma_2 = \sigma_3 \\ 0 & \text{otherwise,} \end{cases}$$

then by simple arguments of symmetry, one can easily obtain the following equalities

$$\max_{x \in \tilde{\Sigma} x \tilde{\Sigma}} \quad \min_{y \in \tilde{\Sigma}} \quad E(x,y) = \frac{1}{4}$$

$$\min_{y \in \tilde{\Sigma}} \quad \max_{x \in \tilde{\Sigma} x \tilde{\Sigma}} \quad E(x,y) = \frac{1}{2} .$$

Applying the theorems together with the lemmas, we obtain the following result.

<u>THEOREM:</u> The mixed extension $\tilde{\Gamma}_{\underline{e}}$ of the finite game $\Gamma_{\underline{e}}$ such that for each $i \epsilon N$ and each $(x_i\ x_{f(i)}) \in X_i \times X_{e(i)}$ the function $E_i(x_{e(i)}, x_{f(i)})$ is linear in $x_{e(i)} \in X_{e(i)}$ ; has the following properties:

a) There is at least one $\underline{e}_m$-simple stable point $\bar{x} \in X_N$ such that:

$$E_i(\bar{x}_i, \bar{x}_{e(i)}, \bar{x}_{f(i)}) \geq v_i(\bar{x}_{f(i)}) = v^i(\bar{x}_{f(i)}) \quad \text{for all } i \epsilon N .$$

b) If for each $x \in X_N$ there is a $y \in X_N$ such that for all $i \epsilon N$ :

$$\max_{\omega_i \epsilon X_i} E_i(\omega_i, y_{e(i)}, x_{f(i)}) = v^i(x_{f(i)})$$

then $\tilde{\Gamma}_{\underline{e}}$ has a least one $\underline{e}^m$-simple stable point $\bar{x} \epsilon X_N$ such that

$$E_i(\bar{x}_i, \bar{x}_{e(i)}, \bar{x}_{f(i)}) \leq v_i(\bar{x}_{f(i)}) = v^i(\bar{x}_{f(i)}) \quad \text{for all } i \epsilon N .$$

c) If for $x \in X_N$ there is a $y \epsilon X_N$ such that for $i \epsilon N$ :

$$\min_{\omega_{e(i)} \epsilon X_{e(i)}} E_i(y_i, \omega_{e(i)}, x_{f(i)}) = v_i(x_{f(i)}) ,$$

and

$$\max_{\omega_i \epsilon X_i} E_i(\omega_i, y_{e(i)}, x_{f(i)}) = v^i(x_{f(i)}) ,$$

then $\tilde{\Gamma}_{\underline{e}}$ has at least one n-person $\underline{e}$-simple stable saddle point.

An analogous result to that expressed in part a) of the above theore be easily obtained by a different technique. Since it is interesting, we such an alternative technique.

THEOREM: The mixed extension $\tilde{\Gamma}_e$ of the finite game $\Gamma_e$ has at least one point $\bar{x}\epsilon X_N$ which satisfies:

$$\min_{z_{e(i)}\epsilon Z_{e(i)}} E_i(\bar{x}_i, z_{e(i)}, \bar{x}_{f(i)}) = \tilde{v}_i(\bar{x}_{f(i)}) = \tilde{v}^i(\bar{x}_{f(i)}) \text{ for all } i\epsilon N$$

where $Z_{e(i)} = \tilde{\Sigma}_{e(i)}$ and $\tilde{v}_i(x_{f(i)})$ and $\tilde{v}^i(x_{f(i)})$ are respectively the maximin and minimax values of the zero-sum two-person game

$$\tilde{\Gamma}_C(x_{f(i)}) = \{X_i, Z_{e(i)}, E_i(x_i, z_{e(i)}, x_{f(i)})\}$$

with $x_{f(i)}\epsilon X_{f(i)}$ .

PROOF: For $i\epsilon N$ , consider the continuous function $C_i$ defined by

$$C_i(x_i, x_{f(i)}) = \min_{z_{e(i)}\epsilon Z_{e(i)}} E_i(x_i, z_{e(i)}, x_{f(i)})$$

in the variable $(x_i, x_{f(i)}) \epsilon X_i \times X_{f(i)}$ . $C_i$ is concave in $x_i\epsilon X_i$ for fixed $x_{f(i)}\epsilon X_{f(i)}$ . By the theorem of Nakaido-Isoda (°) , the game

$\Gamma^{**} = \{X_1, \ldots, X_n, C_1, \ldots, C_n\}$ has an equilibrium point $\bar{x}\epsilon X_N$ ; i.e.,

$$\min_{z_{e(i)}\epsilon Z_{e(i)}} E_i(\bar{x}_i, z_{e(i)}, \bar{x}_{f(i)}) = \tilde{v}_i(x_{f(i)}) \quad \text{for all } i\epsilon N ;$$

With this result and the fact that $\tilde{v}_i(x_{f(i)}) = \tilde{v}^i(x_{f(i)})$ for each game $\tilde{\Gamma}_C(x_{f(i)})$ where $x_{f(i)}\epsilon X_{f(i)}$ by the minimax theorem, the existence of that point is guaranteed. Q.E.D.

The point in the previous theorem can be intuitively interpreted as an $\underline{e}_m$-simple stable point of a partially-cooperative game in the following sense. Each player is guaranteed his least position $(\tilde{v}_i(\bar{x}_{f(i)}))$, even though the behavior of his antagonistic coalition could be concentrated on hurting him, in a cooperative way; if in each instance all the players of his indifferent coalition abide by it.

From an intuitive viewpoint it could be surprising that such a point is an $\underline{e}_m$-simple stable point and conversely. This fact can be easily obtained directly from the corresponding definitions.

We remark that the last conditions in the second and third theorems, such as the corresponding in the subsequent results, express the same results of the theorems when all the sets $f(i)$ are empty and therefore they have not any value.

(o): Nikaido, H., and K. Isoda: Note on noncooperative convex games. Pacific J. Math. 5, 807-815 (1955).

[illegible]

# SIMPLE STABLE POINTS IN TOPOLOGICAL LINEAR SPACES

1. By application of a geometric theorem concerning convex sets presented in [1], Fan in [2] has established under general conditions the existence of an equilibrium point in n-person games on real separated linear topological spaces.

The principal result in the present paper is related to the existence of the simple stable points, introduced in our recent note [4], of n-person games defined on real separated topological vector spaces.

This result will be obtained by application of a method which is essentially that due to Fan in [2]. This method uses a generalization of a theorem due to Fan, concerning convex sets.

As an application of the principal result some results concerning continuous n-person games will be derived.

2. For our purpose, we need a generalized form of Knaster-Kuratowski-Mazurkiewicz's theorem for a real separated topological linear space $Y$ given in [1].

LEMMA 1 (Fan): Let $X$ be a set in a real separated topological vector space $Y$. For each $x \in X$, let $S(x)$ be a closed subset of $Y$, such that:

(i) The convex hull of any finite subset $\{x_1, \dots, x_m\}$ of $X$ is a subset of $\bigcup_{i=1}^{m} S(x_i)$

(ii) For at least one $x \in X$ the set $S(x)$ is compact.

Then $\bigcap_{x \in X} S(x) \neq \emptyset$.

By application of this result we derive the following.

THEOREM 2: Let $X_1, \dots, X_n$ be compact, convex sets, each in a real separated topological vector space. For each $i \in N = \{1, \dots, n\}$, let $h(i)$ be a subset of $N$, and

$$X_{h(i)} = \prod_{j\epsilon h(i)} X_j \; , \; X^{h(i)} = \prod_{j\notin h(i)} X_j .$$

Let $X = \prod_{i=1}^{n} X_i$. For each $x\epsilon X$, let $x_{h(i)}$ be the projection of $x$ in $X_{h(i)}$, and let $x^{h(i)}$ be the projection of $x$ in $X^{h(i)}$. Given $n$ subsets $S_1,\ldots,S_n$ of $X = \prod_{j=1}^{n} X_j$ such that

(i) For each $i\epsilon N = \{1,\ldots,n\}$ and each $x\epsilon X$ the cylinder

$$S_i(x) = \{y\epsilon X\colon (y_{h(i)}, x^{h(i)}) \in S_i\}$$ is convex.

(ii) For each $i\epsilon N$ and $x\epsilon X$ the cylinder

$$S^i(x) = \{y\epsilon X\colon (x_{h(i)}, y^{h(i)}) \in S_i\}$$

is open in X.

(iii) For each $x\epsilon X$ there is a $y\epsilon X$ such that:

$$(y_{h(i)}, x^{h(i)}) \in S_i \quad \text{for all } i\epsilon N .$$

Then $\bigcap_{i=1}^{n} S_i \neq \emptyset$.

<u>PROOF:</u> For each $x\epsilon X$, consider the compact set $A(x)$ defined as the complement in $X$ of the intersection of the $S^i(x)$:

$$A(x) = c\left(\bigcap_{i=1}^{n} S^i(x)\right).$$

By the last condition, the set $\bigcap_{x\epsilon X} A(x)$ is empty; and therefore by the lemma there exists a $z = \sum_{j=1}^{m} \alpha_j \, x(j)$, where $x(j)\epsilon X$, $\alpha_j \geq 0$ and $\sum_{j=1}^{m} \alpha_j = 1$, which does not belong to the set $\bigcup_{j=1}^{m} A(x(j))$. Hence, for each $j \in \{1,\ldots,m\}$ and each $i\epsilon N$: $x(j) \in S_i(z)$: and consequently,

$$z = \sum_{j=1}^{m} \alpha_j \, x(j) \in S_i(z) \qquad \text{for each } i\epsilon N,$$

which implies that $z \in \bigcap_{i=1}^{n} S_i$ . Q.E.D.

A particular case is immediately derived when $h(i) = \{i\}$ for each $i \in N$ . By simplicity we use $x_i$ and $x^i$ for $x_{\{i\}}$ and $x^{\{i\}}$ .

<u>COROLLARY 3 (Fan):</u> Let $X_1,\dots,X_n$ be non-empty compact convex sets each in a real separated topological vector space. Let $S_1,\dots,S_n$ be n ~~subsets~~ of $X$ such that:

(i) For each $i \in N$ and each $x \in X$ the cylinder

$$S_i(x) = \{y \in X: (y_i, x^i) \in S_i\}$$

is non-empty and convex.

(ii) For each $i \in N$ and each $x \in X$ the cylinder

$$S^i(x) = \{y \in X: (x_i, y^i) \in S_i\}$$

is open in $X$ .

Then $\bigcap_{i=1}^{n} S_i \neq \emptyset$ .

<u>PROOF:</u> Since for each $i \in N$ and each $x \in X$ the set $S_i(x)$ is non-empty, we can choose for each $i \in N$ a $y(i) \in S_i(x)$ . Therefore for each $x \in X$ there is an $y \in X$ such that $(y_i, x^i) \in S_i$ for each $i \in N$ , namely $y = (y_1(1),\dots,y_n(n))$ . Consequently, the requirements of the previous theorem are satisfied. Q.E.D.

A real function $f$ defined on a topological space $X$ is said to be lower-semicontinuous (upper-semicontinuous) on $X$ , if for each real number $r$ , the set $\{x \in X: f(x) > r\}$ ($\{x \in X: f(x) < r\}$) is open.

A real function $f$ defined on a convex set of a real vector space $X$ is said to be quasi-concave (quasi-convex) on $X$ , if for each real number $r$ the set $\{x \in X: f(x) > r\}$ ($\{x \in X: f(x) < r\}$) is convex.

THEOREM 4: Let $X_1, \ldots, X_n$ be non-empty, compact, convex sets each in a real separated topological vector space, and let $f_1, \ldots, f_n$ be a real function defined on $X$, having the following properties:

(i) For each $i \in N$ and fixed $x_{h(i)} \in X_{h(i)}$, the function $f_i(x_{h(i)}, x^{h(i)})$ is lower-semicontinuous in $x^{h(i)} \in X^{h(i)}$.

(ii) For each $i \in N$ and fixed $x^{h(i)} \in X^{h(i)}$, the function $f_i(x_{h(i)}, x^{h(i)})$ is quasi-concave in $x_{h(i)} \in X_{h(i)}$.

(iii) Given $r = (r_1, \ldots, r_n)$, for each $x \in X$ there is a $y \in X$ such that $f_i(y_{h(i)}, x^{h(i)}) > r_i$ for every $i \in N$.

Then there exists an $\bar{x} \in X$ such that $f_i(\bar{x}_{h(i)}, \bar{x}^{h(i)}) > r_i$ for all $i \in N$.

PROOF: Consider for each $i \in N$, the set

$$S_i = \{x \in X: f_i(x_{h(i)}, x^{h(i)}) > r_i\}.$$

Then on one hand, the cylinders

$$S_i(x) = \{y \in X: f_i(y_{h(i)}, x^{h(i)}) > r_i\}$$

are convex. On the other hand, the cylinders

$$S^i(x) = \{y \in X: f_i(x_{h(i)}, y^{h(i)}) > r_i\}$$

are open in $X$. Furthermore, for each $x \in X$ there is a $y \in X$ such that $(y_{h(i)}, x^{h(i)}) \in S_i$ for each $i \in N$.

Hence, theorem 2 applied to the sets $S_i$, guarantees the existence of the po[int] $\bar{x} \in X$ such that: $f_i(\bar{x}) > r_i$ for each $i \in N$. Q.E.D.

The condition on the last result of a real valued function is unnecessarily restrictive. Indeed, the result is valid for functions with values in an ordered [s...]

If, for every $i \in N$, $h(i) = \{i\}$, the above result is the same as theorem 2 given in [1].

It is interesting to observe that in general,

$$g_i = \inf_{y^{h(i)}} \sup_{x_{h(i)}} f_i(x_{h(i)}, y^{h(i)}) > r_i$$

for every $i \in N$ does not imply condition (iii) of the last theorem. However, condition (iii) is assured in the simple case where $h(i) = \{i\}$ for each $i \in N$. Therefore, one obtains the following statement (related also in [1]): if, for each $i \in N$, $g_i > r_i$, then there exists an $\bar{x} \in X$ such that $f_i(\bar{x}_i, \bar{x}^i) > r_i$ for every $i \in N$.

3. Now, we consider as applicatons the following theorems concerned with simple stable points of games.

> <u>THEOREM 5:</u> Let $X_1, \ldots, X_n$ be non-empty, compact, convex sets, each in a real separated topological vector space. For each $i \in N = \{1, \ldots, n\}$, let $e(i)$ be a subset of $N - \{i\}$ and $f(i) = N - (e(i) \cup \{i\})$. Let $A_1, \ldots, A_n$ be $n$ continuous real functions defined on $X$, such that for each $i \in N$ and fixed $x_{f(i)} \in X_{f(i)}$ the function $F_i$ defined by
>
> $$F_i(x_i, x_{f(i)}) = \min_{\omega_{e(i)} \in X_{e(i)}} A_i(x_i, \omega_{e(i)}, x_{f(i)}),$$
>
> is quasi-concave with respect to $x_i \in X_i$.
>
> Then there exists an $\bar{x} \in X$ such that
>
> $$F_i(\bar{x}_i, \bar{x}_{f(i)}) = \max_{\omega_i \in X_i} F_i(\omega_i, \bar{x}_{f(i)}) \qquad \text{for every } i \in N.$$

Such a point is a $\underline{e}_m$-simple stable point of the game $\Gamma = \{X_1, \ldots, X_n; A_1, \ldots, A_n\}$.

**<u>PROOF:</u>** For each $i \in N$ and each $\delta > 0$, consider the set

$$S_{\delta,i} = \{x \epsilon X : F_i(x_i, x_{f(i)}) > \max_{\omega_i \epsilon X_i} F_i(\omega_i, x_{f(i)}) - \delta\}$$

Let $h(i) = \{i\}$. Since the functions $F_i$ and $\max\limits_{\omega_i \epsilon X_i} F_i$ are continuous, then the cylinder

$$S^i_\delta(x) = \{y \epsilon X : F_i(x_i, y_{f(i)}) > \max_{\omega_i \epsilon X_i} F_i(\omega_i, y_{f(i)}) - \delta\}$$

is open in $X$. Because the function $F_i$ is quasi-concave in $x_i \epsilon X$, the cylinder

$$S_{\delta,i}(x) = \{y \epsilon X : F_i(y_i, x_{f(i)}) > \max_{\omega_i \epsilon X_i} F_i(\omega_i, x_{f(i)}) - \delta\}$$

is convex. Then by application of corollary 3, we have

$$\bigcap_{i=1}^{n} S_{\delta,i} \neq \emptyset \qquad \text{for every } \delta > 0 ;$$

and therefore there exists a point $\bar{x} \epsilon X$ such that

$$\bar{x} \in \bigcap_{i=1}^{n} \bar{S}_{\delta,i} \qquad \text{for every } \delta > 0 .$$

Such a point satisfies

$$F_i(\bar{x}_i, \bar{x}_{f(i)}) = \max_{\omega_i \epsilon X_i} F_i(\omega_i, \bar{x}_{f(i)}) \quad \text{for every } i \epsilon N . \quad \text{Q.E.D.}$$

We note that this proof is essentially that given in [2] which proves the existence of an equilibrium point. The reason of this connection is that an $\underline{e}_m$-simple stable point of the game $\Gamma = \{X_1, \ldots, X_n, A_1, \ldots, A_n\}$ related in the above theorem, is an equilibrium point of the game $\Gamma^* = \{X_1, \ldots, X_n; F_1, \ldots, F_n\}$, and conversely.

<u>THEOREM 6:</u> Let $X_1, \ldots, X_n$ be non-empty, compact, convex sets each in a real separated topological vector space. For each $i \epsilon N = \{1, \ldots, n\}$, let $e(i)$ be a subset of $N - \{i\}$ and $f(i) = N - (e(i) \cup \{i\})$.

Let $A_1,\dots,A_n$ be $n$ continuous real functions defined on $X$, such that for each $i\epsilon N$ and fixed $x_{f(i)}\epsilon X_{f(i)}$ the function $G_i$ defined by

$$G_i(x_{e(i)},x_{f(i)}) = \max_{\omega_i\epsilon X_i} A_i(\omega_i,x_{e(i)},x_{f(i)}) ,$$

is quasi-convex in $x_{e(i)}\epsilon X_{e(i)}$. If for each $x\epsilon X$, there is a $y\epsilon X$ such that

$$G_i(y_{e(i)},x_{f(i)}) = \min_{\omega_{e(i)}\epsilon X_{e(i)}} G_i(\omega_{e(i)},x_{f(i)}) \quad \text{for every } i\epsilon N ,$$

then there exists a $\bar{x}\epsilon X$ such that

$$G_i(\bar{x}_{e(i)},\bar{x}_{f(i)}) = \min_{\omega_{e(i)}\epsilon X_{e(i)}} G_i(\omega_{e(i)},\bar{x}_{f(i)}) \quad \text{for every } i\epsilon N .$$

Such a point is an $\underline{e}^m$-simple stable point of the game

$$\Gamma = \{X_1,\dots,X_n;\ A_1,\dots,A_n\} .$$

__PROOF:__ The last condition implies the following one: for each $\delta > 0$ and each $x\epsilon X$ there is a $y\epsilon X$ such that

$$G_i(y_{e(i)},x_{f(i)}) < \min_{\omega_{e(i)}\epsilon X_{e(i)}} G_i(\omega_{e(i)},x_{f(i)}) + \delta \qquad \text{for each } i\epsilon N .$$

For each $i\epsilon N$ and each $\delta>0$, consider the set

$$S_{\delta,i} = \{x\epsilon X:\ G_i(x_{e(i)},x_{f(i)}) < \min_{\omega_{e(i)}\epsilon X_{e(i)}} G_i(\omega_{e(i)},x_{f(i)}) + \delta\} .$$

Let $h(i) = e(i)$. Then the cylinder

$$S_{\delta,i}(x) = \{y\epsilon X:\ G_i(y_{e(i)},x_{f(i)}) < \min_{\omega_{e(i)}\epsilon X_{e(i)}} G_i(\omega_{e(i)},x_{f(i)}) + \delta\}$$

is convex, since the function $G_i$ is quasi-convex in $x_{e(i)}\epsilon X_{e(i)}$. Because the functions $G_i$ and $\min_{\omega_{e(i)}\epsilon X_{e(i)}} G_i$ are continuous, the cylinder

$$S^i_\delta(x) = \{y \in X: G_i(x_{e(i)}, y_{f(i)}) < \min_{\omega_{e(i)} \in X_{e(i)}} G_i(\omega_{e(i)}, y_{f(i)}) + \delta\}$$

is open in $X$. Finally, by the last condition, we have that, for each $\delta > 0$ and each $x \in X$, there is a $y \in X$ such that $(y_{e(i)}, x_{f(i) \cup \{i\}}) \in S_{\delta,i}$ for all $i \in N$.

Then, by application of theorem 2 to the sets $S_{\delta,i}$, we have:

$$\bigcap_{i=1}^{n} S_{\delta,i} \neq \emptyset, \qquad \text{for every } \delta > 0 ;$$

and therefore, there exists a point $\bar{x}$:

$$\bar{x} \in \bigcap_{i=1}^{n} \bar{S}_{\delta,i} \qquad \text{for every } \delta > 0 .$$

Such a point satisfies:

$$G_i(\bar{x}_{e(i)}, \bar{x}_{f(i)}) = \min_{\omega_{e(i)} \in X_{e(i)}} G_i(\omega_{e(i)}, \bar{x}_{f(i)}) \qquad \text{for every } i \in N . \quad \text{Q.E.D.}$$

THEOREM 7: Let $X_1, \ldots, X_n$ be non-empty, compact, convex sets, each in a real separated topological vector space. For each $i \in N = \{1,\ldots,n\}$, let $e(i)$ be a subset of $N - \{i\}$ and $f(i) = N - (e(i) \cup \{i\})$. Let $A_1, \ldots$ be $n$ continuous real functions defined on $X$, such that for each $i \in N$ and fixed $x_{f(i)} \in X_{f(i)}$, the function $F_i$ is quasi-concave in $x_i \in X_i$ and the function $G_i$ is quasi-convex in $x_{e(i)} \in X_{e(i)}$.

If, for each $x \in X$, there is a $y \in X$ such that for every $i \in N$:

$$F_i(y_i, x_{f(i)}) = \max_{\omega_i \in X_i} F_i(\omega_i, x_{f(i)})$$

and

$$G_i(y_{e(i)}, x_{f(i)}) = \min_{\omega_{e(i)} \in X_{e(i)}} G_i(\omega_{e(i)}, x_{f(i)}) ,$$

then, there exists an $\bar{x} \in X$ such that, for every $i \in N$

$$A_i(\bar{x}_i,\bar{x}_{e(i)},\bar{x}_{f(i)}) = F_i(\bar{x}_i,\bar{x}_{f(i)})$$
$$= G_i(\bar{x}_{e(i)},\bar{x}_{f(i)})$$

for every $i \in N$ .

Such a point in an e-simple saddle point of the game

$$\Gamma = \{X_1,\dots,X_n ; A_1,\dots, A_n\} .$$

**PROOF:** Suppose that the function $F_i$ is not quasi-concave in the variable $x_i \in X_i$ for fixed $x_{f(i)} \in X_{f(i)}$ . Then, for a real number $\lambda$ and $x_{f(i)} \in X_{f(i)}$ the set

$$F_\lambda = \{x_i \in X_i : F_i(x_i,x_{f(i)}) > \lambda \}$$

is not convex, that is, there exist $\bar{x}_i$, $\tilde{x}_i \in X_i$ such that for some $\mu \in [0,1]$:

$$F_i(\mu \bar{x}_i + (1-\mu)\tilde{x}_i, x_{f(i)}) \leq \lambda .$$

On the other hand at such points, one has:

$$A_i(\bar{x}_i,\omega_{e(i)},x_{f(i)}) > \lambda \quad \text{and} \quad A_i(\tilde{x}_i, \omega_{e(i)},x_{f(i)}) > \lambda$$

for all $\omega_{e(i)} \in X_{e(i)}$ . In particular at the point $\bar{\omega}_{e(i)} \in X_{e(i)}$ for which

$$F_i(\mu \bar{x}_i + (1-\mu)\tilde{x}_i, x_{f(i)}) = A_i(\mu \bar{x}_i + (1-\mu)\tilde{x}_i,\bar{\omega}_{e(i)},x_{f(i)}) \leq \lambda ,$$

we have.

$$A_i(\bar{x}_i, \bar{\omega}_{e(i)}, x_{f(i)}) > \lambda \quad \text{and} \quad A_i(\tilde{x}_i,\bar{\omega}_{e(i)}, x_{f(i)}) > \lambda .$$

This is impossible, since the function $A_i$ is quasi-concave in $x_i \in X_i$ for fixed $(x_{e(i)}, x_{f(i)}) \in X_{e(i)} \times X_{f(i)}$ . Then $F_i$ is quasi-concave in $x_i \in X_i$ for fixed $x_{f(i)} \in X_{f(i)}$ . Similarly, one can easily prove that the function $G_i$ is quasi-convex in $x_{e(i)} \in X_{e(i)}$ for fixed $x_{f(i)} \in X_{f(i)}$ .

By the last theorem, there exists a point $\bar{x} \in X$ such that

$$F_i(\bar{x}_i, \bar{x}_{f(i)}) = \max_{\omega_i \in X_i} F_i(\omega_i, \bar{x}_{f(i)})$$

and

$$G_i(\bar{x}_{e(i)}, \bar{x}_{f(i)}) = \min_{\omega_{e(i)} \in X_{e(i)}} G_i(\omega_{e(i)}, \bar{x}_{f(i)}) \quad \text{for every } i \in N .$$

On the other hand, for each $i \in N$ and each $x_{f(i)} \in X_{f(i)}$, consider the zero-sum two person game

$$\Gamma(x_{f(i)}) = \{Y_i^1, Y_i^2 ; B_i(y_1, y_2)\}$$

where

$$y_1 \in Y_i^1 = X_i , \quad y_2 \in Y_i^2 = X_{e(i)}$$

and

$$B_i(y_1, y_2) = A_i(x_i, x_{e(i)}, x_{f(i)}) .$$

Now, for $j \in \{1,2\}$, let $\bar{e}(j)$ be the set defined by $\bar{e}(j) = k$, where $k \neq j$ and $k \in \{1,2\}$. Then we have $\bar{f}(j) = \emptyset$.

By application of theorem 5 to the game $\Gamma_i(x_{f(i)})$ with the sets $\bar{e}(j)$ for $j \in \{1,2\}$, since the continuous function $B_i$ is quasi-concave in $y_1 \in Y_i^1$ and quasi-convex in $y_2 \in Y_i^2$, the existence of a point $(\tilde{y}_1, \tilde{y}_2) \in Y_i^1 \times Y_i^2$ such that

$$\begin{aligned} B_i(\tilde{y}_1, \tilde{y}_2) &= \max_{y_1} \min_{y_2} B_i(y_1, y_2) \\ &= \min_{y_2} \max_{y_1} B_i(y_1, y_2) , \end{aligned}$$

is guaranteed.

Thus, we have obtained the result that, for each $i \in N$ and each $x_{f(i)} \in X_{f(i)}$

$$\max_{\omega_i \in X_i} \min_{\omega_{e(i)} \in X_{e(i)}} A(\omega_i, \omega_{e(i)}, x_{f(i)}) = \min_{\omega_{e(i)} \in X_{e(i)}} \max_{\omega_i \in X_i} A(\omega_i, \omega_{e(i)}, x_{f(i)})$$

Then, the point $\bar{x} \in X$ obviously satisfies:

$$\begin{aligned} A_i(\bar{x}_i, \bar{x}_{e(i)}, \bar{x}_{f(i)}) &= \max_{\omega_i \in X_i} F_i(\omega_i, \bar{x}_{f(i)}) \\ &= \min_{\omega_{e(i)} \in X_{e(i)}} G_i(\omega_{e(i)}, \bar{x}_{f(i)}) \quad \text{for each } i \in N. \text{ Q.E.D.} \end{aligned}$$

We note that the particular case of theorem 5 applied to the game $\Gamma_i(x_{f(i)})$ in the above proof is a corollary of Sion's minimax theorem found in [1].

4. In this section some applications of the above results to certain kinds of continuous games are considered.

Let $\Sigma$ be a separated compact space. Then the conjugate space $C^*(\Sigma)$ of the Banach space $C(\Sigma)$ of all real continuous functions on $\Sigma$ is a locally convex, separated real topological linear space, with respect to the weak topology induced by $C(\Sigma)$. The set $X$ of regular Borel measures on $\Sigma$ with total measure one is compact and convex in $C^*(\Sigma)$ with respect to the $w^*$ - topology.

By using these facts, we obtain from theorem 5 the following result.

<u>COROLLARY 9:</u> Let $\Gamma = \{\Sigma_1,\dots,\Sigma_n; A_1,\dots,A_n\}$ be a game, where, for each $i \epsilon N$, $\Sigma_i$ is a separated and compact space, $A_i$ is a real continuous function. Then the mixed extension

$$\tilde{\Gamma} = \{X_1,\dots,X_n; E_1,\dots,E_n\},$$

where for each $i \epsilon N$, $X_i$ is the set of regular Borel measures with measure one and the expectation function is defined by

$$E_i(x_1,\dots,x_n) = \int_{\Sigma_1 x \dots x \Sigma_n} A_i \, d(x_1 x \dots x \, x_n)$$

has an $\underline{e}_m$-simple stable point.

<u>PROOF:</u> Consider for each $i \epsilon N$, the multilinear, real function $E_i$ defined on $X_1 x \dots x \, X_n$, which is continuous. Therefore the function $F_i$ defined by

$$F_i(x_i, x_{f(i)}) = \min_{\omega_{e(i)} \epsilon X_{e(i)}} E_i(x_i, \omega_{e(i)}, x_{f(i)})$$

is concave in $x_i \epsilon X_i$, for each fixed $x_{f(i)} \epsilon X_{f(i)}$. By direct application of theorem 5 to the mixed extension game $\tilde{\Gamma}$, the existence of an $\underline{e}_m$-stable point is guaranteed. Q.E.D.

In an analogous way, from the theorem 6, we have:

COROLLARY 10: Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n ; A_1,\ldots,A_n\}$ be a game, where for each $i\in N$ $\Sigma_i$ is a separated and compact space, and $A_i$ is a real continuous function. Let $\tilde{\Gamma} = \{X_1,\ldots,X_n; E_1,\ldots,E_n\}$ be its mixed extension, such that for each $i\in N$ and fixed $(x_i,x_{f(i)})\in X_i \times X_{f(i)}$ the expectation function $E_i$ is linear in the variable $x_{e(i)}\in X_{e(i)}$.

If for each $x \in X$ there is a $y \in X$ such that for every $i\in N$

$$\max_{\omega_i\in X_i} E_i(\omega_i, y_{e(i)}, x_{f(i)}) = \min_{\omega_{e(i)}\in X_{e(i)}} \max_{\omega_i\in X_i} E_i(\omega_i,\omega_{e(i)},x_{f(i)}).$$

Then, the mixed extension $\tilde{\Gamma}$ has an $e^m$-stable point.

Finally, from corollary 8 we immediately obtain:

COROLLARY 11: Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be a game, where for each $i\in N$ $\Sigma_i$ is a separated and compact space and $A_i$ a real continuous function; and let

$$\tilde{\Gamma} = \{X_1,\ldots,X_n; E_1,\ldots,E_n\}$$

be its mixed extension, such that, for each $i\in N$ and fixed $(x_i,x_{f(i)})\in X_i \times X_{f(i)}$, the expectation function $E_i$ is linear in the variable $x_{e(i)}\in X_{e(i)}$.

If for each $x\in X$ there is a $y\in X$ such that, for every $i\in N$ :

$$\max_{\omega_i\in X_i} E_i(\omega_i,y_{e(i)},x_{f(i)}) = \min_{\omega_{e(i)}\in X_{e(i)}} \max_{\omega_i\in X_i} E_i(\omega_i,\omega_{e(i)},x_{f(i)})$$

and

$$\min_{\omega_{e(i)}\in X_{e(i)}} E_i(y_i,\omega_{e(i)},x_{f(i)}) = \max_{\omega_i\in X_i} \min_{\omega_{e(i)}\in X_{e(i)}} E_i(\omega_i,\omega_{e(i)},x_{f(i)}),$$

then the mixed extension $\tilde{\Gamma}$ has an $e$-simple saddle point.

We note that if for each $i\in N$, the set $e(i) = \emptyset$, then corollary 9, which determines the existence of equilibrium point, coincides with the result given by by Glicksberg [3], which has used a general method of fixed points for multivalued functions on locally convex, compact linear topological spaces.

From theorem 5 one can easily derive the following result:

COROLLARY 12: Let $\Gamma = \{\Sigma_1,\dots,\Sigma_n, A_1,\dots,A_n\}$ be a game, where for each $i\in N$, $\Sigma_i$ is a separated and compact space, $A_i$ a real continuous function.

For each $i\in N$, let $Z_{e(i)}$ be the set of regular Borel measures on $\Sigma_{e(i)} = \mathop{X}_{j\in e(i)} \Sigma_j$ with measure one. Then the extension

$$\tilde{\Gamma}^* = \{X_1,\dots,X_n; H_1,\dots,H_n\}$$

where the payoff function $H_i$ of $i\in N$ is defined by

$$H_i(x_i, z_{e(i)}, x_{f(i)}) = \int_{\Sigma_1 \times \Sigma_{e(i)} \times \Sigma_{f(i)}} A_i \, d(x_i \times z_{e(i)} \times x_{f(i)}) ;$$

has a point $\bar{x} \in X$ such that for all $i\in N$:

$$\min_{z_{e(i)}\in Z_{e(i)}} H_i(\bar{x}_i, z_{e(i)}, \bar{x}_{f(i)}) = \max_{\omega_i \in X_i} \min_{z_{e(i)}\in Z_{e(i)}} H_i(\omega_i, z_{e(i)}, \bar{x}_{f(i)})$$

$$= \min_{z_{e(i)}\in Z_{e(i)}} \max_{\omega_i \in X_i} H_i(\omega_i, z_{e(i)}, \bar{x}_{f(i)}) .$$

Further related topics are given in [5].

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Note: e : german letter

# ANOTHER NOTE ON SIMPLE STABLE POINTS IN TOPOLOGICAL LINEAR SPACES

Ezio Marchi

1. In our recent paper [2], we established some results concerning the simple stable points of games defined on separated, convex, compact, real topological linear spaces. We derived these results by using a generalization of a result given by Fan in [1], which is concerned with the intersection of sets with convex sections.

The object of this note is to prove some existence theorems for simple stable points of games given on convex, compact, real topological linear spaces, by using the same idea used by Nikaido-Isoda [3] in order to prove the existence of an equilibrium point.

There is certain similarity between the results expressed in this paper and the respective results obtained by the mentioned technique. However, neither the results obtained here include the other, nor are included in them.

2. For our purpose, we need the basic result introduced in [3], the application of which will give the principal results.

THEOREM 1 (Nikaido-Isoda): Let $\varphi$ be a real function defined on $\Sigma \times \Sigma$, where $\Sigma$ is non-empty, convex and compact in a real topological linear space, such that the following two conditions are fulfilled:

(i) For each $\sigma \in \Sigma$, the functions $\varphi(\sigma,\tau)$ and $\varphi(\tau,\tau)$ are continuous in $\tau \in \Sigma$.

(ii) For each $\tau \in \Sigma$, the function $\varphi(\sigma,\tau)$ is concave in $\sigma \in \Sigma$.

Then there exists a point $\bar{\tau} \in \Sigma$ such that

$$\varphi(\bar{\tau},\bar{\tau}) = \max_{s \in \Sigma} \varphi(s,\bar{\tau})$$

PROOF: Assume that there is not a point having the property just mentioned.

Then, for each $\tau\epsilon\Sigma$, there is a $\sigma\epsilon\Sigma$ such that

$$\varphi(\tau,\tau) < \varphi(\sigma,\tau) .$$

Let

$$\theta_\sigma = \{\tau\epsilon\Sigma:\ \varphi(\tau,\tau) < \varphi(\sigma,\tau)\}$$

be a set in $\Sigma$.

By the continuity of $\varphi(\sigma,\tau)$ and $\varphi(\tau,\tau)$ in $\tau\epsilon\Sigma$ for each $\sigma\epsilon\Sigma$, there ixists a finite number of $\sigma_1,\ldots,\sigma_n \epsilon \Sigma$ such that

$$\bigcup_{i=1}^{n} \theta_{\sigma_i} = \Sigma .$$

Consider the functions

$$\rho_i(\tau) = \max\ [\varphi\ (\sigma_i,\tau) - \varphi(\tau,\tau)\ ,\ 0\ ] \qquad \text{for } i=1,\ldots,n .$$

Therefore by definition

$$\rho(\tau) = \sum_{i=1}^{n} \rho_i(\tau) > 0 \qquad \text{for all } \tau\epsilon\Sigma .$$

Let

$$\psi(\tau) = \sum_{i=1}^{n} \frac{\rho_i(\tau)}{\rho(\tau)}\ \sigma_i ,$$

which defines a function

$$\psi :\ \Sigma \to \Sigma ,$$

since $\Sigma$ is convex.

The convex hull of $\sigma_1,\ldots,\sigma_n$ in $\Sigma$ is homeomorphic to a simplex in an Euclidean space. Then, the application of Brower's fixed point to the function guarantees the existence of a fixed point $\tilde{\tau}$ :

$$\tilde{\tau} = \sum_{i=1}^{n} \frac{\rho_i(\tilde{\tau})}{\rho(\tilde{\tau})}\ \sigma_i .$$

From the last condition we obtain

$$\varphi(\tilde{\tau},\tilde{\tau}) > \varphi(\tilde{\tau},\tilde{\tau})$$

which is impossible. Q.E.D.

3. Let

$$\Gamma = \{\Sigma_1,\ldots,\Sigma_n;\ A_1,\ldots,A_n\}$$

be a n-person game where, for each $i\epsilon N = \{1,\ldots,n\}$ the set $\Sigma_i$ is non-empty, compact, and convex in a real topological linear space and the payoff function $A_i$ is defined on $\Sigma = \underset{i\epsilon N}{X} \Sigma_i$ with values in the real numbers.

Let

$$e(i) \subseteq N - \{i\} \text{ and } f(i) = N - (e(i) \cup \{i\})$$

be the set of players for each $i\epsilon N$, and consider $\Sigma_R = \underset{j\epsilon R}{X} \Sigma_j$ with $R : e(i)$ or $f(i)$.

A point $\bar{\sigma}\epsilon\Sigma$ is said to be a $\underline{e}_m$-simple stable point of the game $\Gamma$ if, for all $i\epsilon N$,

$$\min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\bar{\sigma}_i,s_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_i\epsilon\Sigma_i} \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(s_i,s_{e(i)},\bar{\sigma}_{f(i)}) .$$

Such a point can be easily characterized by the function

$$\Phi_1(\sigma,\tau) = \sum_{i\epsilon N} F_i(\sigma_i,\tau_{f(i)}) ,$$

where, for each $i\epsilon N$, the function $F_i$ is defined by

$$F_i(\sigma_i,\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} A_i(\sigma_i,s_{e(i)}, \sigma_{f(i)}) .$$

<u>LEMMA 2:</u> A point $\bar{\sigma} \in \Sigma$ is a $\underline{e}_m$-simple stable point of the game $\Gamma$ if and only if

$$\Phi_1(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \Phi_1(s,\bar{\sigma}) .$$

<u>PROOF:</u> Let $\bar{\sigma}\in\Sigma$ be a $\underline{e}_m$-simple stable point of the game $\Gamma$. Then, for each ieN,

$$F_i(\bar{\sigma}_i,\bar{\sigma}_{f(i)}) = \max_{s_i\in\Sigma_i} F_i(s_i,\sigma_{f(i)}) ,$$

and therefore,

$$\Phi_1(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \Phi_1(s,\bar{\sigma}) .$$

Now, examine the sufficiency. Let $\bar{\sigma} \in \Sigma$ be such a point which fulfills, for each $\tau \in \Sigma$,

$$\Phi_1(\bar{\sigma},\bar{\sigma}) \geq \Phi_1(\tau,\bar{\sigma}) .$$

Suppose that there is a $\tau \in \Sigma$ and a non-empty subset $I \subseteq N$ such that for each $i\in I$,

$$F_i(\bar{\sigma}_i,\bar{\sigma}_{f(i)}) < F_i(\tau_i,\bar{\sigma}_{f(i)}) .$$

Consider the strategy $\bar{\tau} \in \Sigma$ defined by

$$\bar{\tau} = \begin{cases} \tau_i & \text{if } i\in I \\ \bar{\sigma}_i & \text{if } i\in N-I \end{cases} ,$$

then the following satisfied:

$$\Phi_1(\bar{\sigma},\bar{\sigma}) < \Phi_1(\bar{\tau},\bar{\sigma})$$

which is absurd. Q.E.D.

A point $\bar{\sigma} \in \Sigma$ is said to be a $\underline{e}^m$-simple stable point of the game $\Gamma$ if, for all $i\in N$,

$$\max_{s_i\in\Sigma_i} A_i(s_i,\bar\sigma_{e(i)},\bar\sigma_{f(i)}) = \min_{s_{e(i)}\in\Sigma_{e(i)}} \max_{s_i\in\Sigma_i} A_i(s_i,s_{e(i)},\bar\sigma_{f(i)}) .$$

Introducing for each $i\in N$ the function $G_i$ defined by

$$G_i(\sigma_{e(i)},\sigma_{f(i)}) = \max_{s_i\in\Sigma_i} A_i(s_i,\sigma_{e(i)},\sigma_{f(i)}) ,$$

then it is possible to characterize an $\underline{e}^m$-simple stable point by the function

$$\Phi_2(\sigma,\tau) = \sum_{i\in N} [- G_i (\sigma_{e(i)}, \tau_{f(i)}) ] ,$$

as is illustrated in the following:

__LEMMA 3:__ If for each $\sigma\in\Sigma$ there is a $\tau\in\Sigma$ such that for each $i\in N$ :

$$-G_i(\tau_{e(i)},\sigma_{e(i)}) = \max_{s_{e(i)}\in\Sigma_{e(i)}} [-G_i(s_{e(i)},\sigma_{f(i)})] ,$$

then a point $\bar\sigma\in\Sigma$ is a $\underline{e}^m$-simple stable point of the game $\Gamma$ if and only if

$$\Phi_2(\bar\sigma,\bar\sigma) = \max_{s\in\Sigma} \Phi_2(s,\bar\sigma) .$$

__PROOF:__ Let $\bar\sigma\in\Sigma_N$ be an $\underline{e}_m$-simple stable point of the game $\Gamma$ . Then, for each $i\in N$

$$-G_i(\bar\sigma_{e(i)},\bar\sigma_{f(i)}) = \max_{s_{e(i)}\in\Sigma_{e(i)}} [-G_i(s_{e(i)},\bar\sigma_{f(i)})] .$$

Therefore

$$\sum_{i\in N} [-G_i(\bar\sigma_{e(i)},\bar\sigma_{f(i)})] = \sum_{i\in N} \max_{s_{e(i)}\in\Sigma_{e(i)}} [-G_i(s_{e(i)},\bar\sigma_{f(i)})] ,$$

which implies the validity of the below equality

$$\Phi_2(\bar\sigma,\bar\sigma) = \max_{s\in\Sigma} \Phi_2(s,\bar\sigma) .$$

Now, consider a point $\bar{\sigma}\epsilon\Sigma$ which satisfied

$$\Phi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} \Phi_2(s,\bar{\sigma}) ,$$

and suppose that there is a non-empty subset $I \subseteq N$ such that, for each $i\epsilon I$,

$$-G_i(\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) < \max_{s_{e(i)}\epsilon\Sigma_{e(i)}} [-G_i(s_{e(i)},\bar{\sigma}_{f(i)})] .$$

By hypothesis, given the point $\bar{\sigma}\epsilon\Sigma$, there exists a point $\bar{\tau}\epsilon\Sigma$ such that, for each $i\epsilon N$,

$$-G_i(\bar{\tau}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_{e(i)}\epsilon\Sigma_{e(i)}} [-G_i(s_{e(i)},\bar{\sigma}_{f(i)})] ,$$

and therefore, we obtain

$$\Phi_2(\bar{\sigma},\bar{\sigma}) < \Phi_2(\bar{\tau},\bar{\sigma})$$

which contradicts the hypothesis. Q.E.D.

An immediate consequence of the preceeding lemmas is given in the following result:

<u>COROLLARY 4:</u> If for each $\sigma\epsilon\Sigma$ there is a $\tau\epsilon\Sigma$ such that, for all $i\epsilon N$,

$$F_i(\tau_i,\sigma_{f(i)}) = \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)})$$

and

$$-G_i(\tau_{e(i)},\sigma_{f(i)}) = \max_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) ,$$

then, a point $\bar{\sigma}\epsilon\Sigma$ is an $\underline{e}_m$ and $\underline{e}^m$-simple stable point of the game $\Gamma$ if and only if

$$\Phi_1(\bar{\sigma},\bar{\sigma}) = \max_{s\epsilon\Sigma} \Phi_1(s,\bar{\sigma})$$

and
$$\Phi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \Phi_2(s,\bar{\sigma}) .$$

4. In this section, we will obtain some general theorems which are concerned with the existence of simple stable points of n-person games.

These theorems will be obtained as a direct application of the above results.

**THEOREM 5:** Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be a game, where for each $i\in N$, the set $\Sigma_i$ is convex, compact in a real topological linear space, such that the following conditions are fulfilled:

(i) For each $i\in N$ and each $\sigma_{f(i)}\in\Sigma_{f(i)}$, the function $F_i(\sigma_i,\sigma_{f(i)})$ is concave in $\sigma_i\in\Sigma_i$.

(ii) For each $i\in N$ and each $\sigma_i\in\Sigma_i$, the function $F_i(\sigma_i,\sigma_{f(i)})$ is continuous in $\sigma_{f(i)}\in\Sigma_{f(i)}$.

(iii) The function
$$\sum_{i\in N} F_i(\sigma_i,\sigma_{f(i)})$$
is continuous in $\sigma\in\Sigma$.

Then, there exists an $\underline{e}_m$-simple stable point of the game $\Gamma$.

**PROOF:** Consider the function

$$\Phi_1(\sigma,\tau) = \sum_{i\in N} F_i(\sigma_i,\tau_{f(i)})$$

defined on the set $\Sigma \times \Sigma$. For each $\tau\in\Sigma$, the function $\Phi_1(\sigma,\tau)$ is concave in $\sigma\in\Sigma$. On the other hand, the function $\Phi_1(\sigma,\sigma)$ is continuous in $\sigma\in\Sigma$; and for each $\sigma\in\Sigma$, the function $\Phi_1(\sigma,\tau)$ is continuous in $\tau\in\Sigma$.

Then, by direct application of Theorem 1 to the function $\Phi_1$, the existence of a point $\bar{\sigma}\in\Sigma$ such that

$$\Phi_1(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \Phi_1(s,\bar{\sigma})$$

is guaranteed.

By Lemma 2, such a point is an $\underline{e}_m$-simple stable point of the game $\Gamma$. Q.E.D.

THEOREM 6: Let $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ be a game, where for each $i\in N$, the set $\Sigma_i$ is comvex, compact in a real topological linear space, such that the following conditions are fulfilled:

(i) For each $i\in N$ and fixed $\sigma_{f(i)}\in\Sigma_{f(i)}$, the function

$$G_i(\sigma_{e(i)},\sigma_{f(i)})$$

is convex in $\sigma_{e(i)}\in\Sigma_{e(i)}$.

(ii) For each $i\in N$ and each $\sigma_{e(i)}\in\Sigma_{e(i)}$ the function $G_i(\sigma_{e(i)},\sigma_{f(i)})$ is continuous in $\sigma_{f(i)}\in\Sigma_{f(i)}$.

(iii) The function

$$\sum_{i\in N} G_i(\sigma_{e(i)},\sigma_{f(i)})$$

is continuous in $\sigma\in\Sigma$.

(iv) For each $\sigma\in\Sigma$ there is a $\tau\in\Sigma$ such that for each $i\in N$:

$$-G_i(\tau_{e(i)},\sigma_{f(i)}) = \max_{s_{e(i)}\in\Sigma_{e(i)}} [-G_i(s_{e(i)},\sigma_{f(i)})].$$

Then, there exists an $\underline{e}^m$-simple stable point of the game $\Gamma$.

**PROOF:** Consider the function

$$\Phi_2(\sigma,\tau) = \sum_{i\in N} [-G_i(\sigma_{e(i)},\tau_{f(i)})]$$

defined as $\Sigma \times \Sigma$ . On one hand, for each $\tau\in\Sigma$ , the function $\Phi_2(\sigma,\tau)$ is concave in $\sigma\in\Sigma$ ; and, on the other hand, the function $\Phi_2(\sigma,\sigma)$ is continuous in $\sigma\in\Sigma$ . Furthermore, for each $\sigma\in\Sigma$ , the function $\Phi_2(\sigma,\tau)$ is continous in $\tau\in\Sigma$ .

Then, Theorem 1 applied to the function $\Phi_2$ guarantees the existence of a point $\bar{\sigma}\in\Sigma$ such that

$$\Phi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \Phi_2(s,\bar{\sigma}) .$$

By Lemma 3, such a point is a $\underline{e}^m$-simple stable point of the game $\Gamma$ . Q.E.D.

<u>THEOREM 7:</u> Let $\Gamma = \{\Sigma_1,\ldots, \Sigma_n; A_1,\ldots,A_n\}$ be a game where for each $i\in N$ the set $\Sigma_i$ is convex, compact in a real topological linear space, such that the following conditions are fulfilled:

(i) For each $i\in N$ and each $\sigma_{f(i)}\in\Sigma_{f(i)}$ , the function $F_i(\sigma_i,\sigma_{f(i)})$ is concave in $\sigma_i\in\Sigma_i$ , and the function $G_i(\sigma_{e(i)},\sigma_{f(i)})$ is convex in $\sigma_{e(i)}\in \Sigma_{e(i)}$ .

(ii) For each $i\in N$ and every $\sigma_i\in\Sigma_i$ and $\sigma_{e(i)}\in\Sigma_{e(i)}$ the functions $F_i(\sigma_i,\sigma_{f(i)})$ and $G_i(\sigma_{e(i)}, \sigma_{f(i)})$ are continuous in $\sigma_{f(i)}\in\Sigma_{f(i)}$.

(iii) The functions

$$\sum_{i\in N} F_i(\sigma_i,\sigma_{f(i)}) \quad \text{and} \quad \sum_{i\in N} G_i(\sigma_{e(i)},\sigma_{f(i)})$$

are continuous in $\sigma\in\Sigma$ .

(iv) For each $\sigma\epsilon\Sigma$ there is a $\tau\epsilon\Sigma$ such that for each $i\epsilon N$

$$F_i(\tau_i,\sigma_{f(i)}) = \max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)})$$

and

$$-G_i(\tau_{e(i)},\sigma_{f(i)}) = \max_{s_{e(i)}\epsilon\Sigma_{e(i)}} [-G_i(s_{e(i)},\sigma_{f(i)})] .$$

Then, there exists an $\underline{e}_m$ and $\underline{e}^m$-simple stable point of the game $\Gamma$.

<u>PROOF:</u> Again, consider the function

$$\Phi(\sigma,\tau) = \Phi_1(\sigma,\tau) + \Phi_2(\sigma,\tau)$$

defined as $\Sigma \times \Sigma$. The function $\Phi(\sigma,\sigma)$ is continuous in $\sigma\epsilon\Sigma$ since the functions $\Phi_1(\sigma,\sigma)$ and $\Phi_2(\sigma,\sigma)$ are continuous in $\sigma\epsilon\Sigma$. Moreover, since the functions $\Phi_1(\sigma,\tau)$ and $\Phi_2(\sigma,\tau)$ are continuous in $\tau\epsilon\Sigma$ for each $\sigma\epsilon\Sigma$, then the function $\Phi(\sigma,\tau)$ is continuous in $\tau\epsilon\Sigma$ for each $\sigma\epsilon\Sigma$. Finally, for each $\tau\epsilon\Sigma$, the function $\Phi(\sigma,\tau)$ is concave in $\sigma\epsilon\Sigma$, since it is a sum of concave functions.

Then, Theorem 1 applied to the function $\Phi(\sigma,\tau)$ guarantees the existence of a point $\bar\sigma \in \Sigma$ such that

$$\Phi_1(\bar\sigma,\bar\sigma) + \Phi_2(\bar\sigma,\bar\sigma) = \max_{s\epsilon\Sigma} [\Phi_1(s,\bar\sigma) + \Phi_2(s,\bar\sigma)] .$$

By the last condition, there is a $\bar\tau\epsilon\Sigma$ such that

$$\Phi_1(\bar\tau,\bar\sigma) + \Phi_2(\bar\tau,\bar\sigma) = \max_{s\epsilon\Sigma} [\Phi_1(s,\bar\sigma) + \Phi_2(s,\bar\sigma)] .$$

On the other hand, for each $\tau\epsilon\Sigma$:

$$\Phi_1(\tau,\bar\sigma) \leq \Phi_1(\bar\tau,\bar\sigma) = \sum_{i\epsilon N} \max_{s_i\epsilon\Sigma_i} F_i(s_i,\bar\sigma_{f(i)})$$

and

$$\Phi_2(\tau,\bar\sigma) \leq \Phi_2(\bar\tau,\bar\sigma) = \sum_{i\epsilon N} \max_{s_{e(i)}\epsilon\Sigma_{e(i)}} [-G_i(s_{e(i)},\bar\sigma_{f(i)})]$$

which implies that

$$\Phi_1(\bar{\sigma},\bar{\sigma}) + \Phi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \Phi_1(s,\bar{\sigma}) + \max_{s\in\Sigma} \Phi_2(s,\bar{\sigma}) ,$$

and therefore

$$\Phi_1(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \Phi_1(s,\bar{\sigma})$$

and

$$\Phi_2(\bar{\sigma},\bar{\sigma}) = \max_{s\in\Sigma} \Phi_2(s,\bar{\sigma}) .$$

Then, by corollary 4, such a point is a $\underline{e}_m$ and $\underline{e}^m$-simple stable point of the game $\Gamma$ . Q.E.D.

The above results are the principal of this paper. We note that Theorem 5 essentially coincides with the theorem of Nikaido-Isoda in [3], since a $\underline{e}_m$-simple stable point of a game $\Gamma = \{\Sigma_1,\ldots,\Sigma_n; A_1,\ldots,A_n\}$ is an equilibrium point of the game $\bar{\Gamma} = \{\Sigma_1,\ldots,\Sigma_n; F_1,\ldots,F_n\}$ and conversely.

An immediate consequence of the last theorem is the following:

<u>COROLLARY 8:</u> Let $\Gamma = \{\Sigma_1,\ldots, \Sigma_n; A_1,\ldots,A_n\}$ be a game that satisfies all the conditions of the last theorem. If for each $\sigma\epsilon\Sigma$ and each $i\epsilon N$ ,

$$\max_{s_i\epsilon\Sigma_i} F_i(s_i,\sigma_{f(i)}) = \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\sigma_{f(i)}) ,$$

then there exists an $\underline{e}_m$ and $\underline{e}^m$-simple stable point $\bar{\sigma}\epsilon\Sigma$ such that for each $i\epsilon N$

$$A_i(\bar{\sigma}_i,\bar{\sigma}_{e(i)},\bar{\sigma}_{f(i)}) = \max_{s_i\epsilon\Sigma_i} F_i(s_i,\bar{\sigma}_{f(i)})$$

$$= \min_{s_{e(i)}\epsilon\Sigma_{e(i)}} G_i(s_{e(i)},\bar{\sigma}_{f(i)}) .$$

Such a point is called an $\underline{e}$-simple saddle point of the game $\Gamma$ .

Indeed, there are games for which the additional condition in this last corollary can be in some sense weakened. In fact, this is possible by using Sion's minimax theorem [1] for games defined on separated, real topological linear spaces.

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